Further notes on PMT readout with AC coupling.

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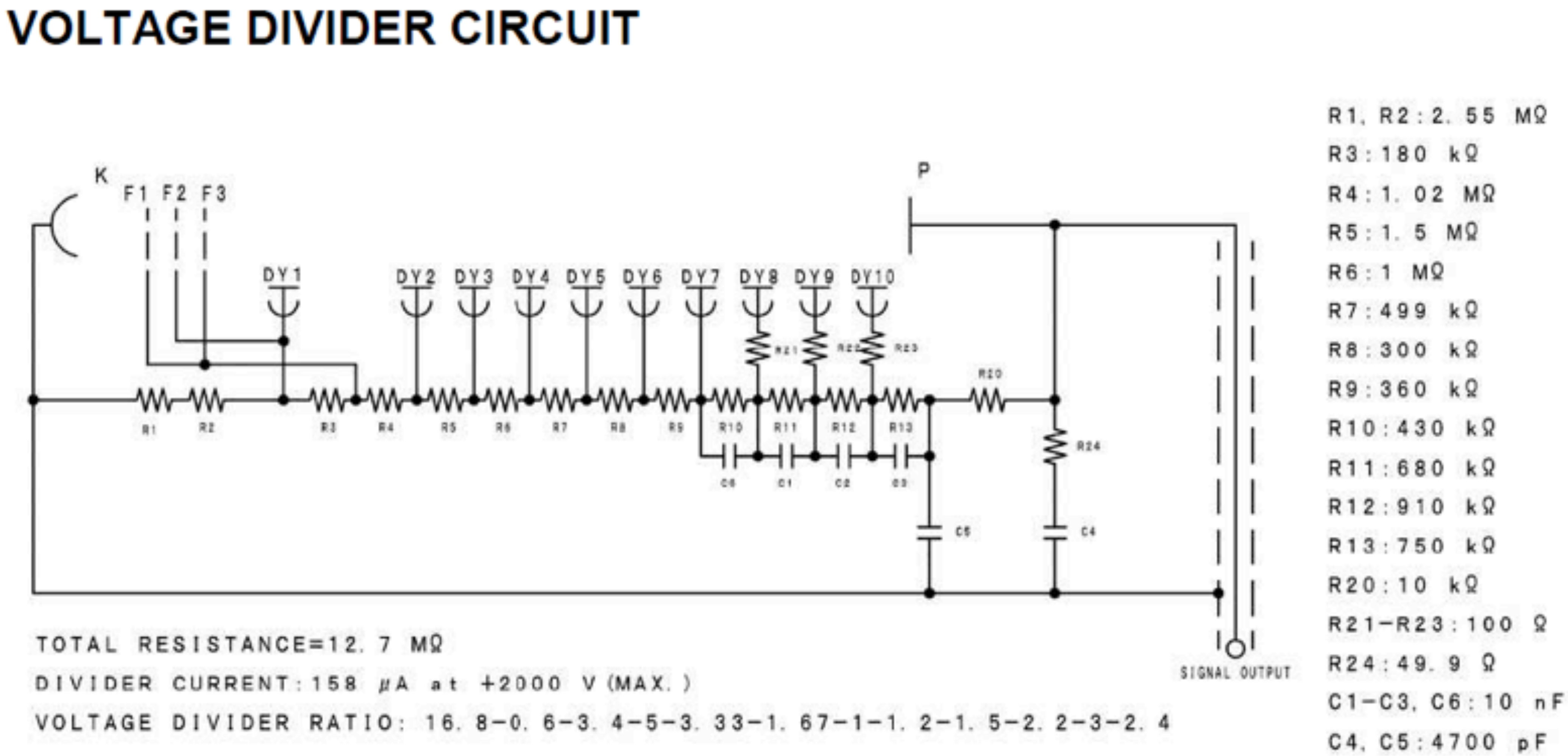
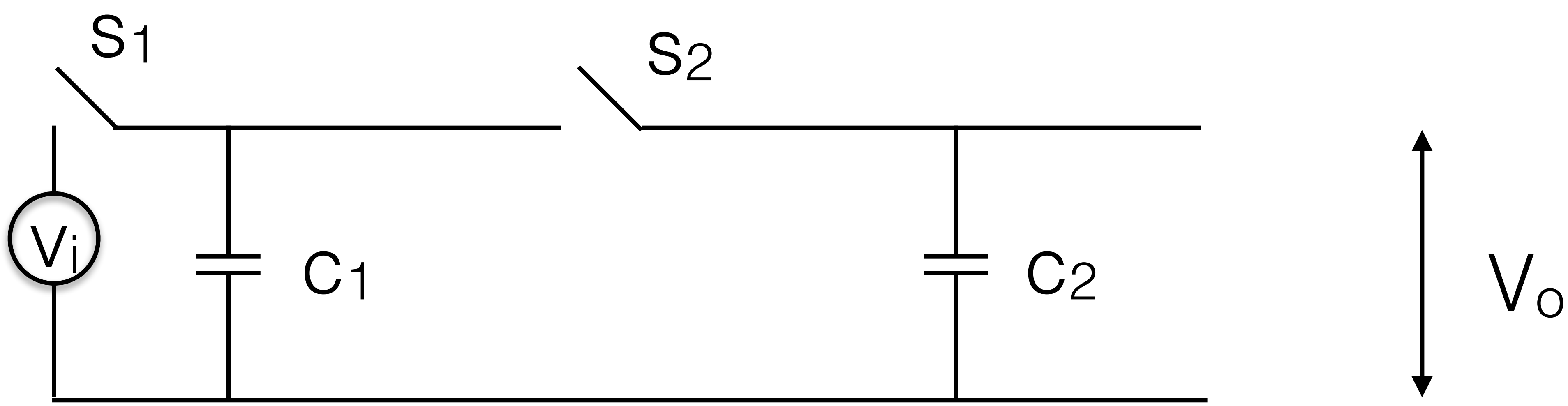
Why ac coupling ?

- ***If the photocathode must be kept at ground potential then the multiplier must be run so that the anode is at positive HV.***
- ***But then the anode current pulse must be decoupled from the high voltage. This has to be done with a capacitor that can handle high voltage.***
- ***An advantage of this scheme is that a single cable can be used to both supply high voltage and carry signal.***
- ***But the disadvantage is that the signal must go through a capacitor and therefore it is differentiated.***

Example from Hamamatsu



This is the divider circuit from HPK for a 10 stage R7081 tube for positive HV. The important items are the large total resistance (12.7 MOhm) to ground, and the capacitors (c4, c5). Clearly a Norton equivalent will have a large resistor and a capacitor.

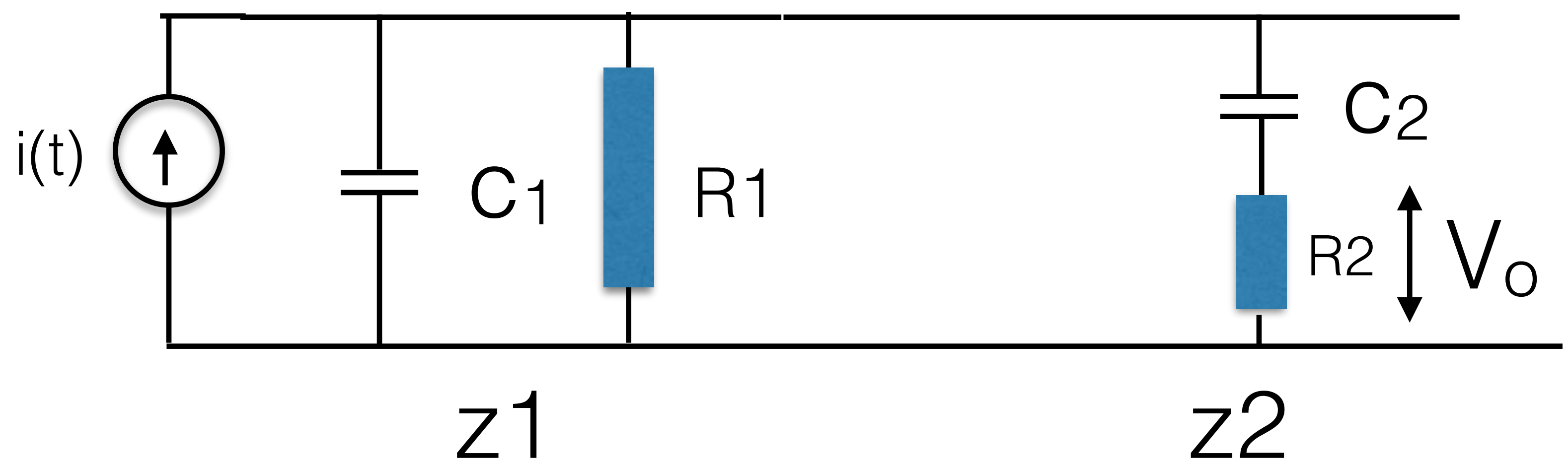


- ***This is a simple arrangement. We examine what happens as switches are closed and opened. Also imagine that there is a long cable between c1 and c2. The cable capacitance is included in c2.***

time	s1	s2	q1	v1	q2	v2	v0
t0	o	o	0	0	0	0	0
t1	c	o	vi*c1	vi	0	0	0
t2	o	o	vi*c1	vi	0	0	0
t3	o	c	vi*c1*c1/ (c1+c2)	vi*c1/ (c1+c2)	vi*c1*c2/ (c1+c2)	vi*c1/ (c1+c2)	vi*c1/ (c1+c2)

As s2 is closed the charge from c1 gets transferred to c2 (as long as c2>>c1). if there are any resistances in parallel to c1 and c2, they will cause the capacitors to discharge with a time constant ~ rc.

Circuit Narrative



We now think about the equivalent circuit for PMT connected to a capacitively coupled readout.

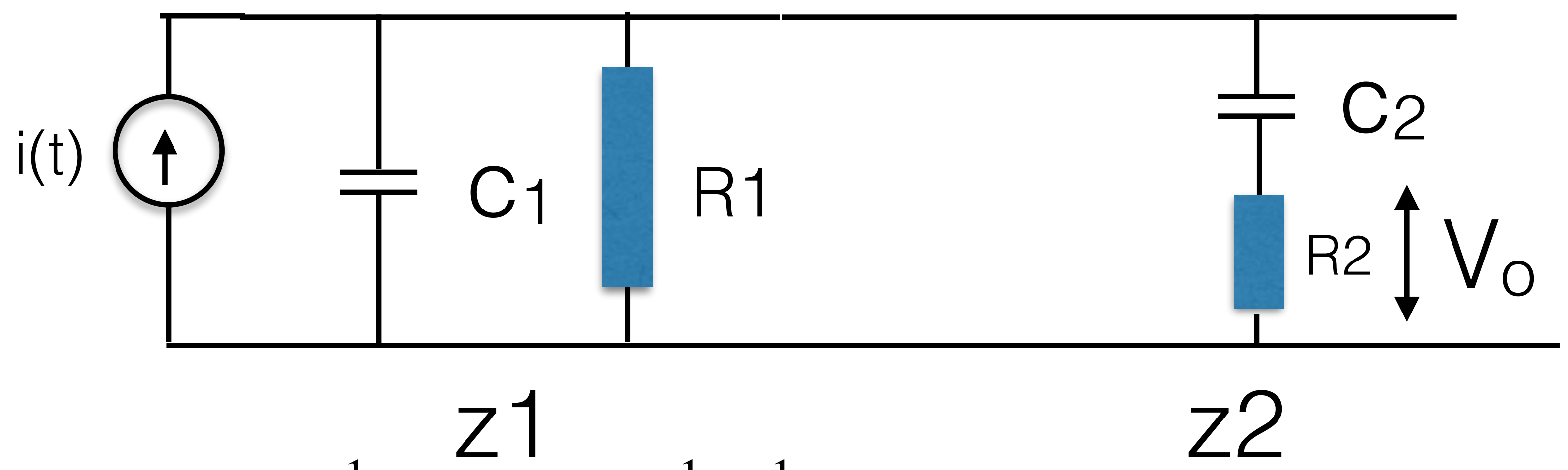
The voltage is read across a resistor R2 (small impedance)

C1 is a small capacitor that gets charged up from the current from the detector. R1 is very large and acts as an effective open switch initially.

The charge from C1 gets transferred to C2 through R2 (with time C1*R2. C2 is very large and has very little voltage buildup.

C2 has to drain over time, and it can only drain through the resistors R1 and R2. Since R1 is very large, it dominates the draining time of R1*C2.

equivalent circuit



$$z_1^{-1} = \frac{1}{r_1} + i\omega c_1, \qquad z_2 = r_2 + \frac{1}{i\omega c_2} \quad \text{and} \quad z^{-1} = \frac{1}{z_1} + \frac{1}{z_2}$$

$$z(\omega) = \frac{r_2 + 1/i\omega c_2}{1 + i\omega c_1 r_2 + c_1/c_2 + r_2/r_1 + 1/i\omega r_1 c_2}$$ notice that care is needed near ω=0

We will remove all undefined quantities from the denominator, but first to get voltage V_0 across r_2 we need to multiply $z(\omega)$ by r_2 / z_2

$$T(\omega) = z(\omega)\frac{r_2}{r_2 + 1/i\omega c_2}$$

$$T(\omega) = r_2 \frac{i\omega r_1 c_2}{1 + (1 + c_1/c_2 + r_2/r_1) i\omega r_1 c_2 - \omega^2 r_1 c_2 r_2 c_1}$$

Now notice that this goes to 0 as $\omega \to 0$ and therefore there is no DC current.

One might wonder what if r_1 were not present. Then the network would not work because any charge that is getting deposited on the capacitors would not be able to discharge over a long time period. Charge from both sides of a capacitor must drain to neutralize the capacitor.

$$V_0(\omega) = I(\omega)T(\omega)$$

We are again going to assume that the incoming current has an exponential pulse with total charge of q_0 and time constant of τ_s

$$T(\omega) = r_2 \frac{i\omega r_1 c_2}{1 + (1 + c_1/c_2 + r_2/r_1) i\omega r_1 c_2 - \omega^2 r_1 c_2 r_2 c_1}$$

use Laplace transform since this is for pulses with $t > 0$

$$V_0(s) = \frac{q_0 r_2}{1 + s\tau_s} \frac{s r_1 c_2}{1 + (1 + c_1/c_2 + r_2/r_1) s r_1 c_2 + s^2 r_1 c_2 r_2 c_1}$$

we can safely assume $c_2 >> c_1$ and $r_1 >> r_2$ and get a much simplified version with two clear time constants. long time constant $\tau_L = r_1 c_2$ and short time constant $\tau_0 = r_2 c_1$

$$V_0(s) = \frac{q_0 r_2}{1 + s\tau_s} \frac{s/\tau_0}{s^2 + s/\tau_0 + 1/(\tau_0 \tau_L)}$$

$$V_0(s) = \frac{q_0 r_2}{\tau_0 \tau_s} \frac{s}{s + 1/\tau_s} \frac{1}{(s + 1/2\tau_0)^2 + (1/(\tau_0 \tau_L) - 1/4\tau_0^2)}$$

we set the second term to $\beta^2 = (1/(\tau_0 \tau_L) - 1/4\tau_0^2)$

With a suitably large τ_L the term is always negative.

$$V_0(s) = \frac{q_0 r_2}{\tau_0 \tau_s} \frac{s}{s + 1/\tau_s} \frac{1}{(s + 1/2\tau_0)^2 + (1/(\tau_0 \tau_L) - 1/4\tau_0^2)}$$

Let's assume that $\beta^2 = (1/(\tau_0 \tau_L) - 1/4\tau_0^2) < 0$

$$V_0(s) = \frac{q_0 r_2}{\tau_0 \tau_s} \frac{s}{s + 1/\tau_s} \times \frac{1}{s + 1/2\tau_0 + \left|1/(\tau_0 \tau_L) - 1/4\tau_0^2\right|^{1/2}} \times \frac{1}{s + 1/2\tau_0 - \left|1/(\tau_0 \tau_L) - 1/4\tau_0^2\right|^{1/2}}$$

$$V_0(s) = \frac{q_0 r_2}{\tau_0 \tau_s} \frac{s}{s + 1/\tau_s} \times \frac{1}{s + 1/\tau_1} \times \frac{1}{s + 1/\tau_2}$$

τ_1 and τ_2 are defined in the obvious way. Notice that $1/\tau_2 > 0$

With some approximations and $\tau_L >> \tau_0$, $1/\tau_1 \approx 1/\tau_0 - 1/\tau_L$ and $1/\tau_2 \approx 1/\tau_L$

and therefore τ_1 is a short time constant and τ_2 is a long time constant.

The solution has three exponentials. We arrange the terms in descending order of time constant

The normalization is such that the initial value at $t = 0$ and the integral $(0 \text{ to } \infty)$ goes to 0.

The resulting pulse will have a single positive root and significant undershoot.

I do not know how to calculate the root exactly.

We will now use this to simulate what happens in case of pileup.

$$v_0(t) = \frac{q_0 r_2}{\tau_s \tau_0}\left[-\frac{e^{-t/\tau_2}}{\tau_2(\frac{1}{\tau_1}-\frac{1}{\tau_2})(\frac{1}{\tau_s}-\frac{1}{\tau_2})} + \frac{e^{-t/\tau_1}}{\tau_1(\frac{1}{\tau_1}-\frac{1}{\tau_2})(\frac{1}{\tau_s}-\frac{1}{\tau_1})} - \frac{e^{-t/\tau_s}}{\tau_s(\frac{1}{\tau_s}-\frac{1}{\tau_2})(\frac{1}{\tau_s}-\frac{1}{\tau_1})} \right]$$

But first let's plot this for some parameters.

table and plots for pmt with ac coupling

Set some parameters

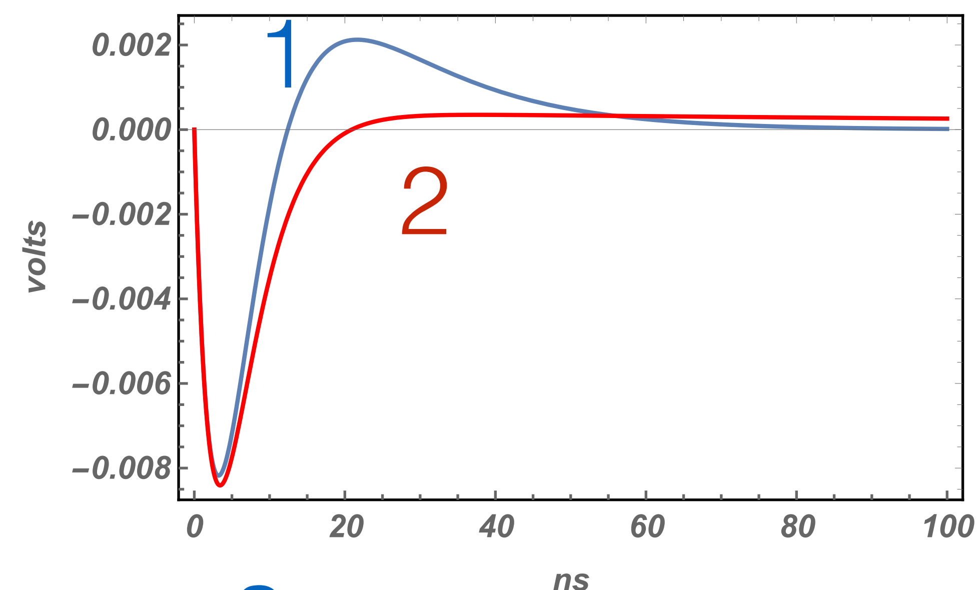
$q_0 = -1.6\times 10^{-12}\,C \qquad r_2 = 50\Omega \qquad \tau_s = 3\times 10^{-9}\,\text{sec}$

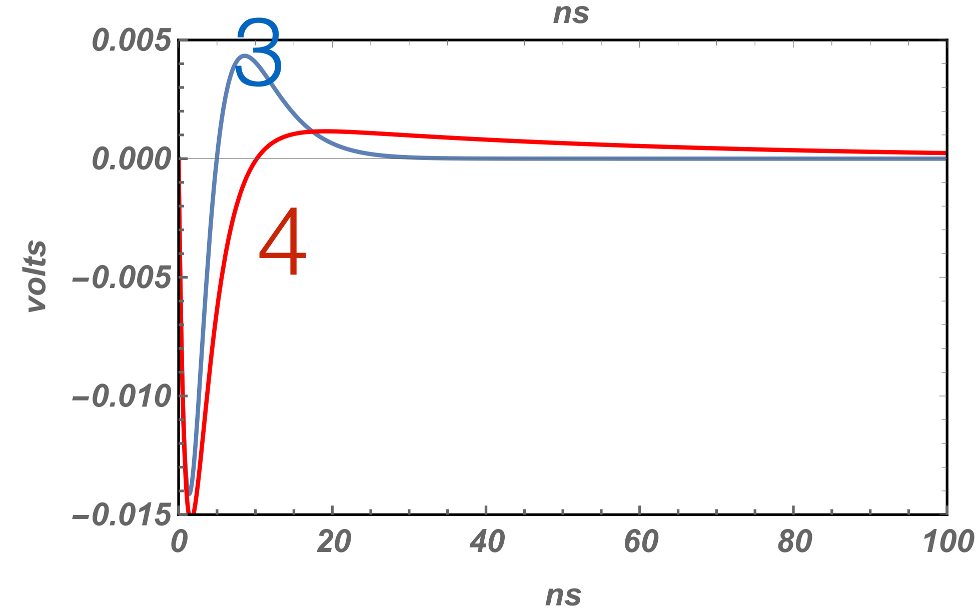
We will vary τ_0 from 1 to 5 ns, and τ_L from 5 to 200 ns.

As can be seen, the pulse can be arranged to have a sharp or long opposite polarity pulse.

In case of long pulse, this will create a DC offset that will depend on the rate of pulses.

	t0	tL	t1	t2
1	4	20	5.5	14.5
2	4	200	4.1	196
3	1	5	1.4	3.6
4	1	50	1.0	49





We will use 1 and 2 for modeling long scintillation pulses

questions

- ***how do we calculate the DC offset ?***
- ***How does the DC offset depend on the rate ?***
- ***what happens if we integrate the pulse with a fixed window ? Does this accurately depict the charge ?***
- ***How do we get the charge which is the normalization of the pulse ?***
- ***How do we get the parameters of the pulse through inspection or through measurement ?***

Some mathematics

Before we start into answering the questions, we need to examine the function. It is truly a fascinating object. The general form has fascinated great mathematicians for centuries.

$$f(t)=\sum_{j=1}^{n} a_j t^{p_j}$$

We will limit ourselves to $a_j, p_j, t \in \mathbb{R};\ t \geq 0$ also $p_1 > p_2 > ... > p_n$

$$e^x = t \quad \rightarrow \quad F(x)=\sum_{j=1}^{n} a_j e^{x p_j} \quad \text{with } x \in \mathbb{R}$$

$$b_j = e^{p_j} \quad \rightarrow \quad F(x)=\sum_{j=1}^{n} a_j b_j^x$$

This is called an exponential polynomial, or generalized polynomial and also a Dirichlet polynomial.
We will use some powerful theorems with simplifications suitable to our 3 term polynomial.

Definitions: assume $f(t)$ has all derivatives, and t_0 is a zero of the function; then if the first non-zero derivative $f^{(k)}(t_0) \neq 0$ is the $k'th$ derivative then t_0 is a zero of the $k'th$ order. If $k=1$ then this is a simple zero.
$Z(F)$ is the sum of all orders of the zeros.
$(a_j)=(a_1, a_2, ..., a_n)$ is the sequence of coefficients in order. $A_j = a_1 + a_2 + ... + a_j$ is a partial sum.

Recall that if p_j are non-negative integers then f(t) is a $n'th$ degree polynomial with at most n zeros. We have methods to find these zeros, but what do we do in the case of a generalized polynomial?

Theorems and references

This turns out to be an important advanced topic in analysis. We are going to just apply some lessons from the following theorems, somewhat sloppily stated (to be brief). These theorems are similar to Descarte's rule of signs.

Th1: For $F(x)$ (as defined on previous slide) with the terms ordered according to $p_1 > p_2 > ... > p_n$, the sum of orders of all positive zeros is not greater than $S[(A_j)]$ which is the number of sign changes in the sequence A_j. The interval $(0,\infty)$ does not include 0.

The theorem can be extended to negatives by reversing the sequence $\mathrm{B}_j = a_{n-j} + .. + a_n$

Th2: With $A_n = 0$, the function has a somewhat special form with $F(0) = 0$, now sum of orders of all zeros $Z(F) \ < \ S[(A_j)] + 1$

References: See a review by Jameson (Math. Gassett 90, No. 518 (2006)) 223-234
Also J. F. Ritt (Trans. Amer. Math. Soc 31 (1929) 680-686.

analysis of $v_0(t)$

First we are going to arrange the function in descending order, taking care to make all terms expressed in the time constants positive.

$$v_0(t) = \frac{q_0 r_2}{\tau_s \tau_0} \frac{-e^{-t/\tau_s}}{\tau_s(\frac{1}{\tau_s} - \frac{1}{\tau_2})(\frac{1}{\tau_s} - \frac{1}{\tau_1})} \left[\frac{\tau_s(\frac{1}{\tau_s} - \frac{1}{\tau_1})}{\tau_2(\frac{1}{\tau_1} - \frac{1}{\tau_2})} e^{t(1/\tau_s - 1/\tau_2)} - \frac{\tau_s(\frac{1}{\tau_s} - \frac{1}{\tau_2})}{\tau_1(\frac{1}{\tau_1} - \frac{1}{\tau_2})} e^{t(1/\tau_s - 1/\tau_1)} + 1 \right]$$

The normalization is such that the initial value at $t = 0$ and the integral $(0 \text{ to } \infty)$ goes to 0. We ignore q_0 and its sign in the following discussion. Assume q_0 is positive. I have pulled the fastest decaying component out with a negative sign.

Now notice the value at ∞ goes to 0 from negative, since it is dominated by the first term in bracket. By construction $A_1 = \frac{\tau_s}{\tau_2}(\frac{1}{\tau_s} - \frac{1}{\tau_1})(\frac{1}{\tau_1} - \frac{1}{\tau_2})^{-1} > 0,\ A_2 = -1 \ and \ A_3 = 0$ and therefore there are two sign changes, with one of them at 0 and second one on the positive side. Both first order.

We can find the second zero by using $\tau_2 >> \tau_1 > \tau_s$ and some successive approximations.

location of zero crossing

We will now approximately find the zero crossing (t_{zero}) for the function in the brackets.

$$f_{bracket}(t) = \frac{\tau_s(\frac{1}{\tau_s} - \frac{1}{\tau_1})}{\tau_2(\frac{1}{\tau_1} - \frac{1}{\tau_2})} e^{t(1/\tau_s - 1/\tau_2)} - \frac{\tau_s(\frac{1}{\tau_s} - \frac{1}{\tau_2})}{\tau_1(\frac{1}{\tau_1} - \frac{1}{\tau_2})} e^{t(1/\tau_s - 1/\tau_1)} + 1$$

The t_{zero} can be bounded from below and above using some observations. It must be greater than the maximum of the function (or the zero of the first derivative !). This is easily obtained.

$$t_{z-} = \frac{Log(\tau_2 / \tau_1)}{(1/\tau_1 - 1/\tau_2)}$$

If τ_2 is large then this is a good approximation.

Upper bound is found by making the observation that the zero of $f_{bracket}(t) - 1$ must be greater than t_{zero}

$$t_{z+} = \frac{Log(\frac{\tau_2(1/\tau_s - 1/\tau_2)}{\tau_1(1/\tau_s - 1/\tau_1)})}{(1/\tau_1 - 1/\tau_2)}$$

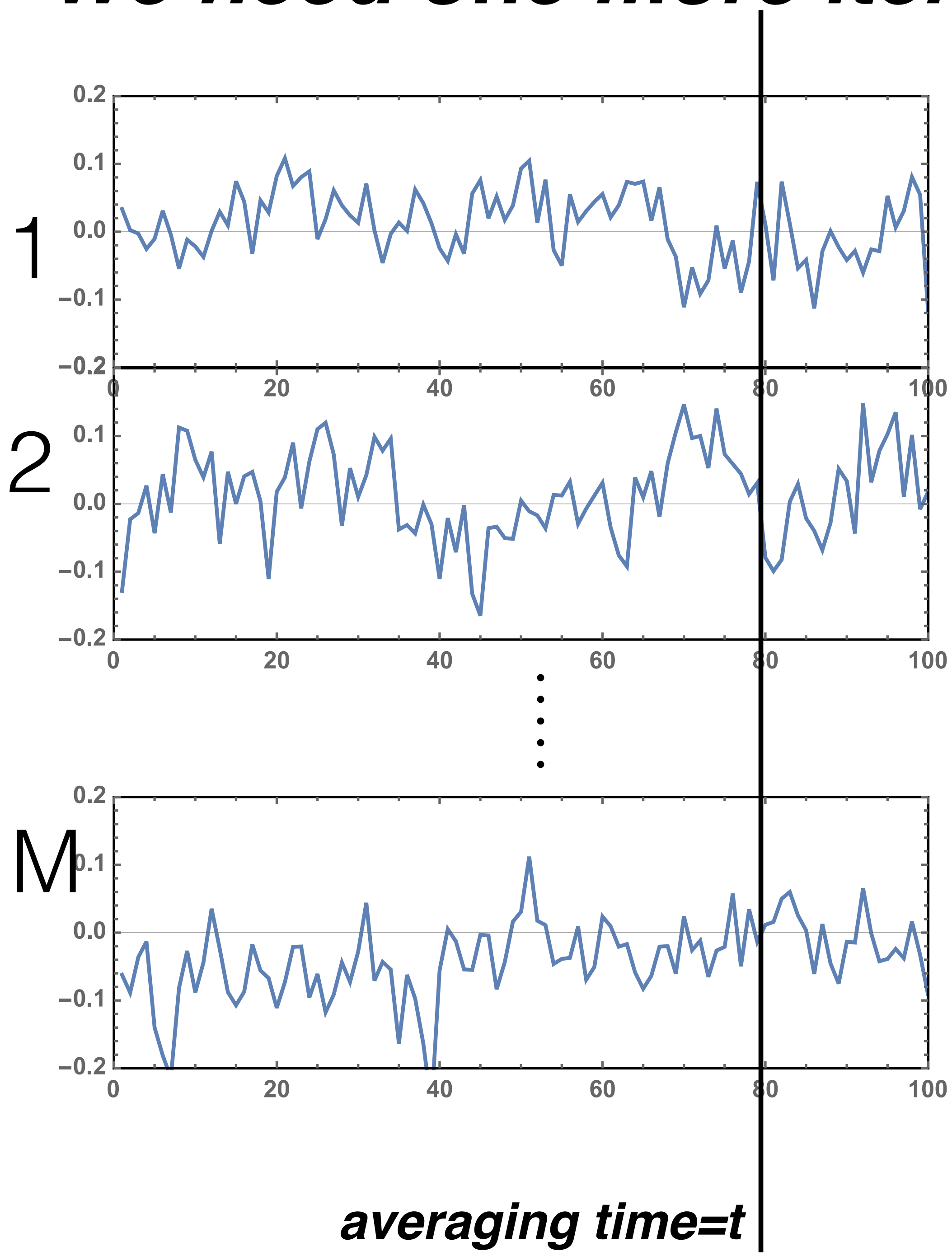
$$t_{z-} < t_{zero} < t_{z+}$$

The approximation can be improved iteratively by using function itself.

$$t_{zero} \approx (1/\tau_2 - 1/\tau_1)^{-1} \times Log\left[\frac{\tau_1(1/\tau_s - 1/\tau_1)}{\tau_2(1/\tau_s - 1/\tau_2)} + \frac{\tau_1(1/\tau_1 - 1/\tau_2)}{\tau_s(1/\tau_s - 1/\tau_2)} e^{-t_{z+}(1/\tau_s - 1/\tau_2)}\right]$$

We are now going to use this with our examples to calculate various quanties.

We need one more item to understand rate fluctuations



The average value of $v_0(t)$ after the zero crossing is what we want.

This is not an average over time ($0 \rightarrow T$), but an average over great many intervals of length T with t held fixed. Imagine that v(t) is a pulse much shorter than T, and rate r =N/T constant.

Then let the number of intervals go to infinity.

We are integrating only above the zero crossing.

$$\langle eg(t)\rangle = r\langle q\rangle \int_{t_z}^{\infty} v(t)\,dt$$

This is the average value of the pulse multiplied by the rate.

The variance of the baseline is

$$\langle v(t)^2\rangle - \langle v(t)\rangle^2 = r\langle q^2\rangle \int_{t_z}^{\infty} v(t)^2\,dt$$

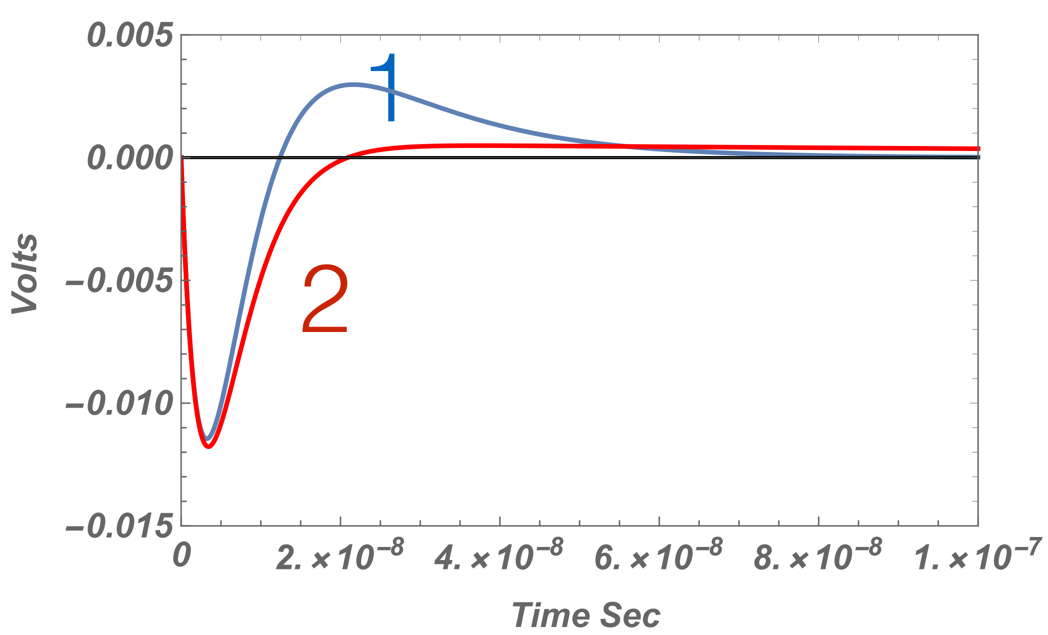
This proof takes a bit of work, ...

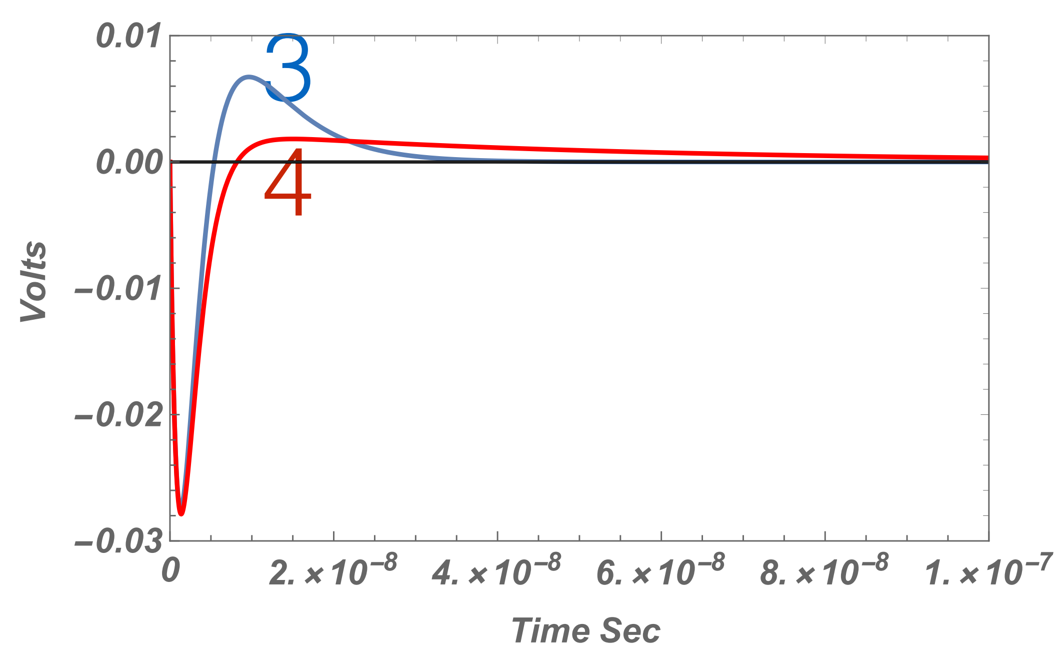
This is called Campbell's theorem. You can learn more from one of my other lectures.

table with zero crossing and charge integration

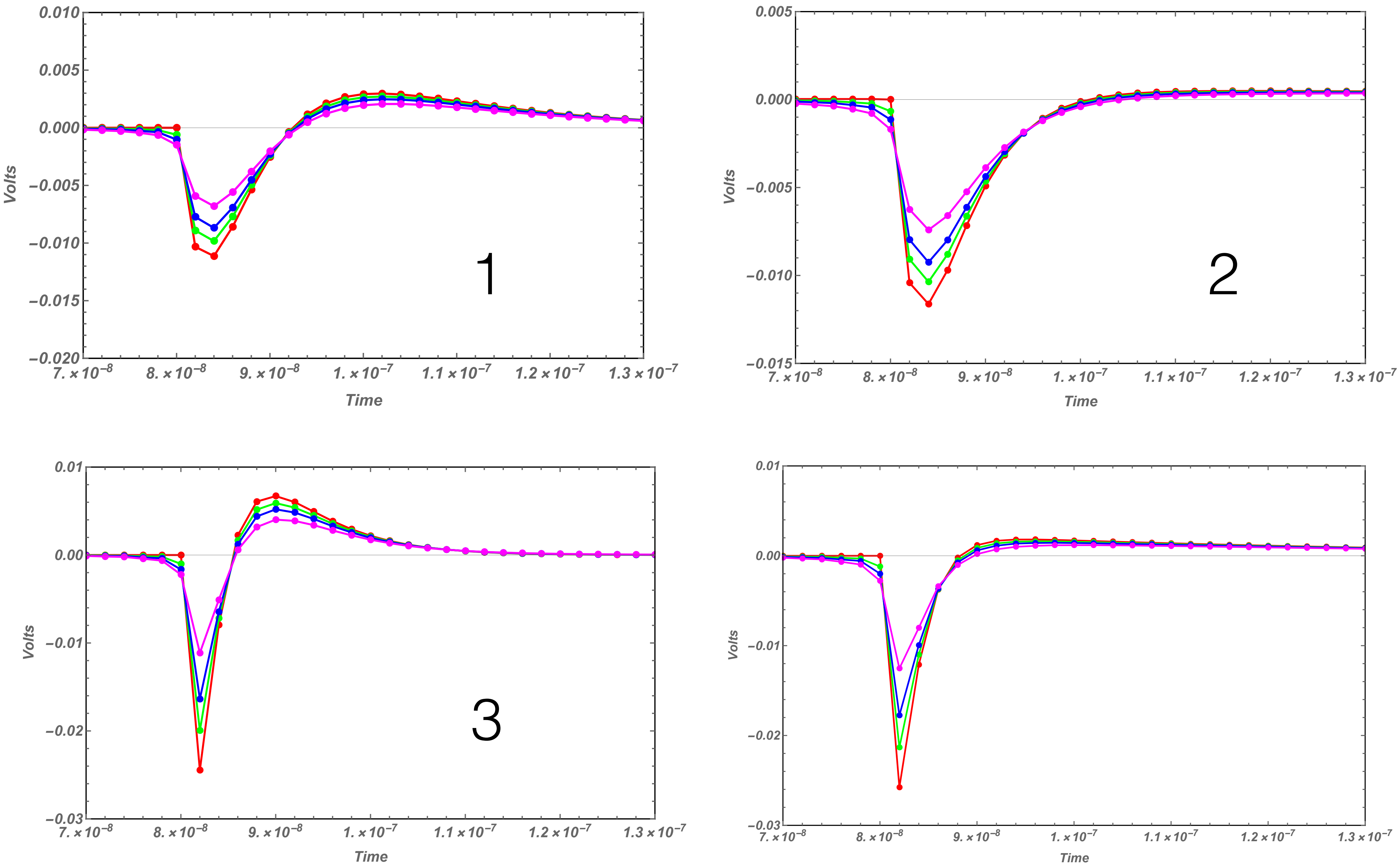
$q_0 = -1.6 \times 10^{-12} C \quad r_2 = 50\Omega$ We will now keep $\tau_s < \tau_0 < \tau_L / 4$ because of the approximations made.

	ts (ns)	t0 (ns)	tL(ns)	t1(ns)	t2(ns)	t-zc(ns)	charge fraction in peak	baseline shift 10 MHz (mV)	baseline fluctuation (mV)
1	3	4	20	5.5	14.5	12.6	0.71	0.80	1.6
2	3	4	200	4.1	196	21.0	0.93	1.04	0.52
3	1	2	9	3	6	5.4	0.72	0.81	1.91
4	1	2	50	1.0	49	8.2	0.90	1.00	1.00





What happens after 0, 20, 40, 80 m of RG59/u ?



We are using the cable attenuation constant of 4.082×10^{-7} per Sqrt[Hz] per meter; see the previous lecture on pmts and cables.

Conclusion

- ***We calculated the pulse shape for an AC coupled network that is commonly used for PMTs. It is characterized by 3 time constants and has a general form.***

- ***The pulse has a zero crossing and the total charge integrates to zero. The current in the long pulse can cause a baseline shift and baseline variance.***

- ***We have formulated a method for obtaining the zero crossing from the time constants.***

- ***If the the 3rd time constant is short then the baseline shift is smaller, but fluctuations are larger.***

- ***The proper way to integrate the charge is up to the zero crossing. The fraction of charge that is in the positive pulse depends on the time constants. In particular, the long time constant.***

- ***If the pulse is digitized then a fit could be performed to get the constants and find the charge. Digitization time needs to be of the same order as the shorter time constant.***

- ***In case of long cables, the pulse shape will broaden and the zero crossing will move farther; in this case, we must develop a way to fit the pulse to obtain the appropriate parameters. The cable will cause fast signals to lose amplitude (but not total charge which gets spread out.).***

- ***The cable will increase the baseline shift because it redistributes the charge to lower frequencies, but reduce the fluctuation. (Prove this to yourselves)***